FMB with rock compressibility - Saturation Equations

Suppose we have a reservoir with pressure-dependent rock compressibility $c_r(p)$, so that its pore volume changes with pressure. By definition

$$c_r(p) = \frac{1}{V_p}\frac{dV_p}{dp} \tag{1}$$

Separating variables and integrating we have

$$\int_{V_p}^{V_{pi}} \frac{dV_p}{V_p} = \int_{p}^{p_i} c_r(p)\,dp \tag{2}$$

or

$$\ln\left(\frac{V_{pi}}{V_p}\right) = \int_{p}^{p_i} c_r(p)\,dp \tag{3}$$

or

$$V_p = V_{pi}\exp\left(-\int_{p}^{p_i} c_r(p)\,dp\right) \tag{4}$$

Define the pore volume multiplier γ as

$$\gamma = \exp\left(-\int_{p}^{p_i} c_r(p)\,dp\right) \tag{5}$$

Our material balance equations become

$$\begin{aligned} M_o(t) &= 5.615\rho_{osc}N_p \\ &= 5.615\rho_{osc}V_{pi}\left(\zeta_{oi} - \gamma\left(\frac{\bar{S}_o}{\bar{B}_o} + \frac{\bar{r}_v\bar{S}_g}{\bar{B}_g}\right)\right) \end{aligned} \tag{6}$$

$$\begin{aligned} M_g(t) &= 1000\rho_{gsc}G_p \\ &= 1000\rho_{gsc}V_{pi}\left(\zeta_{gi} - \gamma\left(\frac{\bar{R}_s\bar{S}_o}{\bar{B}_o} + \frac{\bar{S}_g}{\bar{B}_g}\right)\right) \end{aligned} \tag{7}$$

$$\begin{aligned} M_w(t) &= 5.615\rho_{wsc}W_p \\ &= 5.615\rho_{gsc}V_{pi}\left(\zeta_{wi} - \gamma\frac{\bar{S}_w}{\bar{B}_w}\right) \end{aligned} \tag{8}$$

where

$$\zeta_{oi} \equiv \frac{S_{oi}}{B_{oi}} + \frac{r_{vi}S_{gi}}{B_{gi}} \tag{9}$$

$$\zeta_{gi} \equiv \frac{R_{si}S_{oi}}{B_{oi}} + \frac{S_{gi}}{B_{gi}} \tag{10}$$

and

$$\zeta_{wi} \equiv \frac{S_{wi}}{B_{wi}} \tag{11}$$

and the total mass withdrawal at any time is given by

$$M_T(t) = M_o(t) + M_g(t) + M_w(t) \tag{12}$$

The cumulative fluid ratio relationships become

$$\mathrm{OGR}(t) = \frac{\zeta_{oi} - \gamma\left[\frac{S_o}{B_o} + \frac{r_v S_g}{B_g}\right]_{well}}{\zeta_{gi} - \gamma\left[\frac{R_s S_o}{B_o} + \frac{S_g}{B_g}\right]_{well}} \tag{13}$$

$$\mathrm{WGR}(t) = \frac{\zeta_{wi} - \gamma\left[\frac{S_w}{B_w}\right]_{well}}{\zeta_{gi} - \gamma\left[\frac{R_s S_o}{B} + \frac{S_g}{B_g}\right]_{well}} \tag{14}$$

Solving Eqs. 13 and 14 for the phase saturations, we have

$$S_o = -\frac{B_o\left(\Gamma_o + \zeta_{oi}\left(B_g + B_w\mathrm{WGR}\right) - \zeta_{gi}\left(B_g\mathrm{OGR} + B_w\mathrm{WGR}r_v\right)\right)}{\gamma\left(B_g\left(\mathrm{OGR}R_s - 1\right) - B_o\left(\mathrm{OGR} - r_v\right) + B_w\mathrm{WGR}\left(R_s r_v - 1\right)\right)} \tag{15}$$

where

$$\Gamma_o = \left(\gamma - \zeta_{wi}B_w\right)\left(\mathrm{OGR} - r_v\right) \tag{16}$$

and

$$S_w = -\frac{B_w\left(\Gamma_w + \zeta_{wi}\left(B_g\left(1 - \mathrm{OGR}R_s\right) + B_o\left(\mathrm{OGR} - r_v\right)\right)\right)}{\gamma\left(B_g\left(\mathrm{OGR}R_s - 1\right) - B_o\left(\mathrm{OGR} - r_v\right) + B_w\mathrm{WGR}\left(R_s r_v - 1\right)\right)} \tag{17}$$

where

$$\Gamma_w = \left(\gamma\left(1 - R_s r_v\right) + \zeta_{gi}\left(B_o r_v - B_g\right) + \zeta_{oi}\left(B_g R_s - B_o\right)\right)\mathrm{WGR}$$

The MP FMB Equation is then

$$\frac{\beta_i - \gamma\beta_w}{\dot{m}} = \frac{1}{V_{pi}}\mathrm{MBT}_M + \frac{1}{b_M} \tag{18}$$

Average Pressure Calculation

If we use Eqs. 15 and 17 evaluated at average reservoir pressure in any of the component mass balance equations (Eqs. 6 - 8) we will obtain an implicit equation in average pressure that can be solved iteratively. For the oil equation, material balance can be written as

$$N_i - N_p - \gamma V_{pi}\left(\frac{\bar{S}_o}{\bar{B}_o} + \frac{\bar{r}_v \bar{S}_g}{\bar{B}_g}\right) = 0 \tag{19}$$

Substituting in the saturation equations and setting the numerator of the resulting equation to zero, we obtain after considerable algebraic manipulation

$$\begin{aligned}&\left(N_p - \gamma N_i - (1-\gamma)\, G_i \mathrm{OGR}\right)\left(B_o - B_g R_s\right) + \left(G_p - G_i\right)\left(B_g - B_o r_v\right) \\ &\quad + \left(W_p B_w - (1-\gamma)\, G_i \mathrm{WGR} B_w + \gamma\left(V_{pi} - W_i B_w\right)\right)\left(1 - R_s r_v\right) = 0\end{aligned} \tag{20}$$

where it is understood that all PVT properties are evaluated at the average pressure.

Quadratic Gas Rate Prediction Formula

If we consider 1-day timesteps, the cumulative producing Oil-Gas Ratio (OGR) at day $n+1$ is given by:

$$\mathrm{OGR}^{n+1} = \frac{N_p^{n+1}}{G_p^{n+1}} = \frac{N_p^n + \mathrm{ogr}^{n+1} q_g^{n+1}}{G_p^n + q_g^{n+1}} = \frac{\zeta_{oi} - \gamma\left[\frac{S_o^{n+1}}{B_o^{n+1}} + \frac{r_v^{n+1} S_g^{n+1}}{B_g}\right]}{\zeta_{gi} - \gamma\left[\frac{R_s^{n+1} S_o^{n+1}}{B_o^{n+1}} + \frac{S_g^{n+1}}{B_g^{n+1}}\right]} \tag{21}$$

where ogr^{n+1} is the instantaneous producing oil-gas ratio at time t^{n+1}. Similarly, the cumulative producing Water-Gas Ratio at at day $n+1$ is given by:

$$\mathrm{WGR}^{n+1} = \frac{W_p^{n+1}}{G_p^{n+1}} = \frac{W_p^n + \mathrm{wgr}^{n+1} q_g^{n+1}}{G_p^n + q_g^{n+1}} = \frac{\zeta_{wi} - \gamma\left[\frac{S_w^{n+1}}{B_w^{n+1}}\right]}{\zeta_{gi} - \gamma\left[\frac{R_s^{n+1} S_o^{n+1}}{B_o^{n+1}} + \frac{S_g^{n+1}}{B_g^{n+1}}\right]} \tag{22}$$

where wgr^{n+1} is the instantaneous producing water-gas ratio at time t^{n+1}. In Eqs. 21 and 22 it is understood that all PVT properties and saturations are evaluated at sandface conditions. Using the saturation constraint

$$S_o^{n+1} + S_g^{n+1} + S_w^{n+1} = 1 \tag{23}$$

we can solve Eqs. 21 and 22 for oil and gas saturations in terms of the new gas rate as

$$S_o^{n+1} = \frac{A_o q_g^{n+1} + C_o}{A_d q_g^{n+1} + C_d} \tag{24}$$

and

$$S_w^{n+1} = \frac{A_w q_g^{n+1} + C_w}{A_d q_g^{n+1} + C_d} \tag{25}$$

where

$$\begin{aligned}A_o \quad = \quad & B_o^{n+1}\left\{\left(B_g^{n+1}\mathrm{ogr}^{n+1} + B_w^{n+1} r_v^{n+1}\mathrm{wgr}^{n+1}\right)\zeta_{gi}\right. \\ & - \left(B_g^{n+1} + B_w^{n+1}\mathrm{wgr}^{n+1}\right)\zeta_{oi} \\ & \left. + \left(ogr^{n+1} - r_v^{n+1}\right)\left(B_w^{n+1}\zeta_{wi} - \gamma\right)\right\}\end{aligned} \tag{26}$$

$$
\begin{aligned}
C_o \;=\;& B_o^{n+1}\left\{\left(B_g^{n+1}N_p^n + B_w^{n+1}r_v^{n+1}W_p^n\right)\zeta_{gi}\right. \\
& -\left(B_g^{n+1}G_p^n + B_w^{n+1}W_p^n\right)\zeta_{oi} \\
& \left. +\left(N_p^n - G_p^n r_v^{n+1}\right)\left(B_w^{n+1}\zeta_{wi}-\gamma\right)\right\} \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
A_d \;=\;& \gamma\left(B_g^{n+1}\left(\mathrm{ogr}^{n+1}R_s^{n+1}-1\right)\right. \\
& \left. +B_o^{n+1}\left(r_v^{n+1}-\mathrm{ogr}^{n+1}\right)+B_w^{n+1}\left(R_s^{n+1}r_v^{n+1}-1\right)\mathrm{wgr}^{n+1}\right) \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
C_d \;=\;& \gamma\left(N_p^n\left(B_g^{n+1}R_s^{n+1}-B_o^{n+1}\right)\right. \\
& \left. +G_p^n\left(B_o^{n+1}r_v^{n+1}-B_g^{n+1}\right)+B_w^{n+1}\left(R_s^{n+1}r_v^{n+1}-1\right)W_p^n\right) \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
A_w \;=\;& B_w^{n+1}\times \\
& \left\{\left(\left(B_g^{n+1}-B_o^{n+1}r_v^{n+1}\right)\zeta_{gi}\right.\right. \\
& +\left(B_o^{n+1}-B_g^{n+1}R_s^{n+1}\right)\zeta_{oi}-\gamma\left(1-R_s^{n+1}r_v^{n+1}\right)\Big)\,\mathrm{wgr}^{n+1} \\
& \left. +\left(\mathrm{ogr}^{n+1}\left(B_g^{n+1}R_s^{n+1}-B_o^{n+1}\right)+B_o^{n+1}r_v^{n+1}-B_g^{n+1}\right)\zeta_{wi}\right\} \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
C_w \;=\;& B_w^{n+1}\left\{\left(\gamma\left(R_s^{n+1}r_v^{n+1}-1\right)\right.\right. \\
& \left. -\left(B_o^{n+1}r_v^{n+1}-B_g^{n+1}\right)\zeta_{gi}-\left(B_g^{n+1}R_s^{n+1}-B_o^{n+1}\right)\zeta_{oi}\right)W_p^n + \\
& \left. \left(B_g^{n+1}\left(N_p^nR_s^{n+1}-G_p^n\right)-B_o^{n+1}\left(N_p^n-G_p^nr_v^{n+1}\right)\right)\zeta_{wi}\right\} \qquad (31)
\end{aligned}
$$

Suppose that for a given reservoir we have a well established multiphase flowing material balance straight line,

$$
\frac{\beta_i-\gamma\beta_w^{n+1}}{\dot{m}^{n+1}} = \frac{1}{V_p}MBT_M + \frac{1}{b_M} \qquad (32)
$$

Multiplying both sides of the equation evaluated at time t^{n+1} by the mass flow rate, we have

$$
\beta_i-\gamma\beta_w^{n+1} = \frac{1}{V_p}M_T^{n+1} + \frac{\dot{m}^{n+1}}{b_M} \qquad (33)
$$

The mass flow rate at time t^{n+1} is given by

$$
\dot{m}^{n+1} = q_g^{n+1}\left(5.615\rho_{osc}ogr^{n+1} + 1000\rho_{gsc} + 5.615\rho_{wsc}ogr^{n+1}\right) \qquad (34)
$$

and the cumulative mass at time t^{n+1} is

$$
M_T^{n+1} = M_T^n + \dot{m}^{n+1} \qquad (35)
$$

Let

$$
\alpha_o = 5.615\rho_{osc} \qquad (36)
$$

$$
\alpha_g = 1000\rho_{gsc} \qquad (37)
$$

$$\alpha_w = 5.615\rho_{wsc} \tag{38}$$

$$\alpha_c = \left(\alpha_o ogr^{n+1} + \alpha_g + \alpha_w ogr^{n+1}\right) \tag{39}$$

Eq. 33 can be written as

$$\beta_i - \frac{\gamma\left(D_q q_g^{n+1} + E\right)}{B_g^{n+1} B_o^{n+1} B_w^{n+1}\left(A_d q_g^{n+1} + C_d\right)} = \frac{M_T^n}{V_p} + \alpha_c\left(\frac{1}{V_p} + \frac{1}{b_M}\right) q_g^{n+1} \tag{40}$$

where

$$\begin{aligned} D_q &= A_o B_g^{n+1} B_w^{n+1}\left(\alpha_o + \alpha_g R_s^{n+1}\right) + \alpha_w A_w B_g^{n+1} B_o^{n+1} \\ &\quad + \left(A_d - A_o - A_w\right) B_o^{n+1} B_w^{n+1}\left(\alpha_g + \alpha_o r_v^{n+1}\right) \end{aligned} \tag{41}$$

and

$$\begin{aligned} E &= B_g^{n+1}\left(\alpha_w B_o^{n+1} C_w + \left(\alpha_o + \alpha_g R_s^{n+1}\right) B_w^{n+1} C_o\right) \\ &\quad + \left(\alpha_g + \alpha_o r_v^{n+1}\right) B_o^{n+1} B_w^{n+1}\left(C_d - C_o - C_w\right) \end{aligned} \tag{42}$$

Eq. 40 can be rearranged as a quadratic equation in q_g^{n+1}, i.e.,

$$a\left(q_g^{n+1}\right)^2 + b q_g^{n+1} + c = 0 \tag{43}$$

where

$$a = -\frac{A_d \alpha_c B_g^{n+1} B_o^{n+1} B_w^{n+1}\left(b_M + V_{pi}\right)}{b_M V_{pi}} \tag{44}$$

$$\begin{aligned} b &= \gamma\left(\left(\left(A_o + A_w - A_d\right) B_w^{n+1}\left(\alpha_g + \alpha_o r_v^{n+1}\right) - \alpha_w A_w B_g^{n+1}\right) B_o^{n+1}\right. \\ &\quad \left. - A_o B_g^{n+1} B_w^{n+1}\left(\alpha_o + \alpha_g R_s^{n+1}\right)\right) \\ &\quad - B_g^{n+1} B_o^{n+1} B_w^{n+1}\left(\frac{\left(\alpha_c C_d + A_d M_T^n\right)}{V_{pi}} + \frac{\left(\alpha_c C_d - A_d \beta_i b_M\right)}{b_M}\right) \end{aligned} \tag{45}$$

and

$$\begin{aligned} c &= -\gamma\left(\left(B_w^{n+1} C_o\left(\alpha_o + \alpha_g R_s^{n+1}\right) + \alpha_w B_o^{n+1} C_w\right) B_g^{n+1}\right. \\ &\quad \left. + B_o^{n+1} B_w^{n+1}\left(C_d - C_o - C_w\right)\left(\alpha_g + \alpha_o r_v^{n+1}\right)\right) \\ &\quad + B_g^{n+1} B_o^{n+1} B_w^{n+1} C_d \frac{\left(\beta_i V_{pi} - M_T^n\right)}{V_{pi}} \end{aligned} \tag{46}$$

The positive root of this quadratic is the new predicted rate.

Recovery Factor Calculation

If we consider the average pressure formula, Eq. 20, we have

$$\begin{aligned} &\left(N_p - \gamma N_i - (1-\gamma) G_i \mathrm{OGR}\right)\left(B_o - B_g R_s\right) + \left(G_p - G_i\right)\left(B_g - B_o r_v\right) \\ &\quad + \left(W_p B_w - (1-\gamma) G_i \mathrm{WGR} B_w + \gamma\left(V_{pi} - W_i B_w\right)\right)\left(1 - R_s r_v\right) = 0 \end{aligned} \tag{47}$$

Define

$$RF_o = \frac{N_p}{N_i} \tag{48}$$

$$m = \frac{(G_i - N_i R_{si})\, B_{gi}}{(N_i - G_i r_{vi})\, B_{oi}} = \frac{S_{gi}}{S_{oi}} \tag{49}$$

so

$$\frac{G_i}{N_i} = \frac{m B_{oi} + B_{gi} R_{si}}{B_{gi} + m B_{oi} r_{vi}} \tag{50}$$

The saturation constraint can be written as

$$1 - S_{wi} - S_{oi}\left(1 + \frac{S_{gi}}{S_{oi}}\right) = 1 - S_{wi} - S_{oi}\,(1+m) = 0 \tag{51}$$

so

$$S_{oi} = \frac{(1 - S_{wi})}{(1+m)} \tag{52}$$

Note that

$$\frac{G_p}{N_i} = \frac{G_p}{N_p}\frac{N_p}{N_i} = \frac{RF_o}{\mathrm{OGR}} \tag{53}$$

$$\frac{W_p}{N_i} = \frac{W_p}{G_p}\frac{G_p}{N_i} = \frac{\mathrm{WGR}\,RF_o}{\mathrm{OGR}} \tag{54}$$

$$\begin{aligned}\frac{W_i}{N_i} &= \frac{W_i B_{wi}}{(N_i - G_i r_{vi})\, B_{oi}} \frac{(N_i - G_i r_{vi})\, B_{oi}}{N_i B_{wi}} \\ &= \frac{S_{wi}}{S_{oi}} \frac{\left(1 - \frac{G_i}{N_i} r_{vi}\right) B_{oi}}{B_{wi}}\end{aligned} \tag{55}$$

$$\begin{aligned}\frac{V_p}{N_i} &= \frac{V_p}{(N_i - G_i r_{vi})\, B_{oi}} \frac{(N_i - G_i r_{vi})\, B_{oi}}{N_i} \\ &= \frac{1}{S_{oi}}\left(1 - \frac{G_i}{N_i} r_{vi}\right) B_{oi}\end{aligned} \tag{56}$$

Dividing Eq. 47 by N_i and using the relationships in Eqs. 48 - 56, we have

$$\begin{aligned}&\left(RF_o - \gamma - (1-\gamma)\frac{m B_{oi}}{B_{gi}}\mathrm{OGR}\right)(B_o - B_g R_s) + \left(\frac{RF_o}{\mathrm{OGR}} - \frac{m B_{oi}}{B_{gi}}\right)(B_g - B_o r_v) \\ &+ \left(\frac{\mathrm{WGR}\,RF_o}{\mathrm{OGR}} B_w - (1-\gamma)\frac{m B_{oi}}{B_{gi}}\mathrm{WGR}\,B_w + \gamma\frac{(1+m)\, B_{oi}}{(1-S_{wi})}\left(1 - \frac{S_{wi}}{B_{wi}} B_w\right)\right)(1 - R_s r_v) = 0\end{aligned} \tag{57}$$